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KRDB Research Centre Technical Report:

Introduction to part-whole relations: mereology, conceptual modelling and mathematical aspects

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Introduction to part-whole relations: mereology, conceptual modelling and mathematical aspects*

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1 Introduction

An active area of research in knowledge representation concerns the nature of the part-whole relation between entities or instances, its representation in a (formal) language, and its translation to support in software systems. The sub-disciplines it traverses include philosophy, (applied) ontology and ontologies, linguistics, conceptual modelling, formal languages, and software design & implementation. Depending on one's background, points of view on importance of the parthood relation ranges from essential and equally important on a par with the taxonomic subsumption relation to that there is no real added value. The latter opinion tends to be due either to misunderstanding of the nature of the parthood relation and its variations or that part-whole relations can be accommodated for with set theory anyway (which, as will become clear below, is not always true). In this tutorial, we highlight some of the main aspects and research trends regarding the parthood relation and point toward several open problems awaiting a solution.

The first part consists of philosophical aspects (called mereology) and several mathematical properties of theories of parthood (§2), after which we introduce the contentious notion of types of parthood relations in §3. This is followed by representation of part-whole relations in several conceptual modelling and ontology languages like Description Logics, UML, ER and ORM (§4). A brief discussion of arguments on representing part-whole relations in knowledge representation languages, or not, is included in §5. The last section (§6) summarizes and provides several suggestions for further research. Due to space and time constraints, this tutorial is necessarily incomplete, therefore you are strongly advised to consult the references, which go into detail of particular sub-topics that are covered only briefly here.

2 Mereology and mathematical properties of mereological theories

Mereology, the ontological investigation into the part-whole relation, mainly dates back to Leśniewski [23] in the early 20th century. From the 1980s up till now this research has greatly expanded, with important publications by Peter Simons [37] and Achille Varzi [44], among others. Varzi [44] provides an overview of the more and

* This technical report contains background information and further references accompanying the tutorial that bears the same title as this TR, held at Free University Bozen-Bolzano, Italy, in October 2006.

less constrained versions of mereology from the viewpoint of philosophy, which is summarized by Guizzardi [12] from the perspective of conceptual modelling. Here, we first summarize the main basic aspects of mereology as described by [44] in §2.1, which is augmented with a comparison with set theory as elaborated on by [30] in §2.2. Afterward, some other extensions, variations and gaps pass the revue in §2.3, including initial steps from the philosophical aspects toward computation and toward applicability to specific subject domains.

2.1 Basic principles up to GEM

The very lowest common denominator concerning the parthood relation, called *Ground Mereology* and abbreviated as **M**, is that it is a relation capturing a partial order that is always reflexive (1), antisymmetric (2), and transitive (3) and all other versions share at least these constraints[1]. This, however, does not mean (1-3) are uncontested; in particular transitivity of the *part_of* receives attention, to which we return later.

$$part\_of(x, x) \tag{1}$$

// everything is part of itself

$$(part\_of(x, y) \land part\_of(y, x)) \rightarrow x = y \tag{2}$$

// two distinct things cannot be part of each other, or:
// if x is part of y and y is part of x, then x and y are the same thing

$$(part\_of(x, y) \land part\_of(y, z)) \rightarrow part\_of(x, z) \tag{3}$$

// if x is part of y and y is part of z, then x is part of z

With these three basic formulas that take *part_of* as primitive relation (i.e., it does not have a definition), several other mereological predicates can be built. A common one is the definition of *proper* part as (4), from which asymmetry, and irreflexivity follows; thus, x is not part of itself, if x is part of y then y is not part of x, and if x is part of y and y part of z then x is part of z. Note that in some mereological theories, proper part is taken as primitive relation.

$$proper\_part\_of(x, y) \triangleq part\_of(x, y) \land \neg part\_of(y, x) \tag{4}$$

Six more predicates can be introduced, of which overlap (5) and overcross (7) tend to be more often mentioned, although underlap (6) is more often used in e.g. bio-ontologies.

$$overlap(x, y) \triangleq \exists z (part\_of(z, x) \land part\_of(z, y)) \tag{5}$$

// x and y 'share' a piece z (see also overcross); is reflexive and symmetric

$$underlap(x, y) \triangleq \exists z (part\_of(x, z) \land part\_of(y, z)) \tag{6}$$

[1] All formulas are universally quantified, unless otherwise specified.

// x and y are both part of some z; is reflexive and symmetric

$$overcross(x, y) \triangleq overlap(x, y) \wedge \neg part\_of(x, y) \tag{7}$$

// x and y overlap, but x is not a part of y

$$undercross(x, y) \triangleq underlap(x, y) \wedge \neg part\_of(y, x) \tag{8}$$

// x and y underlap, but y is not part of x

$$proper\_overlap(x, y) \triangleq overcross(x, y) \wedge overcross(y, x) \tag{9}$$

// combines proper parthood with overlap

$$proper\_underlap(x, y) \triangleq undercross(x, y) \wedge undercross(y, x) \tag{10}$$

// combines proper parthood with underlap

Now we can start creating extensions. The first one has to do with the argument that if some y has a proper part x, then there should be some remainder because x is 'less' than y. There are two ways to add this to **M**: through weak (11) and strong (12) supplementation.

$$proper\_part\_of(x, y) \rightarrow \exists z(part\_of(z, y) \wedge \neg overlap(z, x)) \tag{11}$$

// weak supplementation: every proper part must be supplemented by another, disjoint, part

$$\neg part\_of(y, x) \rightarrow \exists z(part\_of(z, y) \wedge \neg overlap(z, x)) \tag{12}$$

// strong supplementation: if an object fails to include another among its parts,
// then there must be a remainder

The addition of the weak supplementation principle to the ground mereology is called *Minimal Mereology*, abbreviated as **MM**, whereas the addition of strong supplementation is called *Extensional Mereology*, or **EM**. However, **EM** has some issues: because (13) is a theorem of **EM**, it follows that non-atomic objects with the same proper parts are identical (14) – but sameness of parts may not be sufficient for identity. For instance, two objects can be distinct solely with respect to how their parts are arranged, and one can make the distinction (or not) between an object and the matter constituting it, like a vase and the clay[2]. Consult [44] section 3.2 for examples, discussion, and pointers to further literature.

$$\begin{aligned} &\exists z(proper\_part\_of(z, x) \rightarrow \\ &\quad (\forall z(proper\_part\_of(z, x) \rightarrow proper\_part\_of(z, y)) \rightarrow part\_of(x, y))) \end{aligned} \tag{13}$$

$$\begin{aligned} &(\exists z(proper\_part\_of(z, x)) \vee \exists z(proper\_part\_of(z, y))) \rightarrow \\ &\quad (x = y \leftrightarrow \forall z(proper\_part\_of(z, x) \leftrightarrow proper\_part\_of(z, y)) \end{aligned} \tag{14}$$

Extending **M** in another direction, we look at the feature that a mereological domain must be closed under various operations. There are two options: finitary operations,

[2] With the multiplicative approach as taken by DOLCE [24], the vase and the amount of clay are not identical because they each have different properties. They have, however, the same parts.

which adds a **C** of *closure*, or unrestricted fusions, which adds the **G** for *general* mereology, which we introduce now.

First, we have two operations: the mereological sum (also called *fusion*) and the product. Given underlap (6), then there is a smallest entity z such that x and y are only and fully part of z (15); likewise for overlap (5), there is a largest common entity that is part of both x and y (16). For instance, a sugar pot as the mereological sum of the cup and the lid, and the junction is the product of two intersecting roads.

$$underlap(x, y) \rightarrow \exists z \forall w(overlap(w, x) \leftrightarrow (overlap(w, x) \vee overlap(w, y))) \quad (15)$$

// mereological sum (fusion)

$$overlap(x, y) \rightarrow \exists z \forall w(part\_of(w, z) \leftrightarrow (part\_of(w, x) \wedge part\_of(w, y))) \quad (16)$$

// mereological product

Adding (15) and (16) to **M** gives the **CM**, and adding it to **EM** gives **CEM**. **CEM** supports definitions for $x + y$ and $x \times y$, which can be succinctly rewritten into (17) and (18) for unique mereological sum of two underlapping entities and unique product of two overlapping entities, respectively.

$$underlap(x, y) \rightarrow \exists z(z = x + y) \quad (17)$$

$$overlap(x, y) \rightarrow \exists z(z = x \times y) \quad (18)$$

Similarly, we can add more closure postulates, like *remainder*, *complementation*, and *top*. Adding bottom, on the other hand, is rarely done – except for good algebraic reasons (see §2.2) – because it means postulating a null entity that is part of everything, i.e. $\exists z \forall x(part\_of(z, x))$, which is philosophically problematic.

Second, we take a look at unrestricted fusions, which will get us to **GEM**. Unrestricted fusions, i.e. sums of arbitrary non-empty sets of objects (and consequently also products) would need explicit reference to classes, by which we have to leave first order theory. However, there is an alternative for permitting unrestricted fusions yet stay within first order, which is by relying on an axiom schema with only predicates or open formulas. Let ϕ be a property or condition, then for every satisfied ϕ there is an entity consisting of all entities that satisfy ϕ (note: since there is a countable amount of open formulas, there are countable many classes). Thus, unrestricted fusion (19) can be added to **M** to give **GM**, which is known as *Classical Mereology* or *General Mereology.*

$$\exists x \phi \rightarrow \exists z \forall y(overlap(y, z) \leftrightarrow \exists x(\phi \wedge overlap(y, x))) \quad (19)$$

// unrestricted fusion

Adding unrestricted fusion (19) to **EM** or an extensional strengthening of **GM** with strong supplementation (12), then we get the *General Extensional Mereology* **GEM**. The various possible mereological theories and how they relate are depicted in Fig.1; a more comprehensive diagram is included as figure 2 in [43].

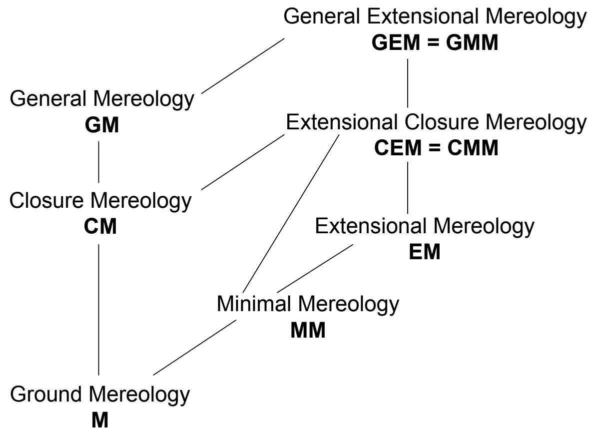

Fig. 1: Hasse diagram of mereological theories; from weaker to stronger, going uphill (after [44]).

We can define the sum σ and product π in **GEM**, which enables one to succinctly rewrite sum (20), product (21), remainder (22), complement (23), and universal individual (24). See [44] sections 4.2 and 4.3 for further detail and discussion.

$$x + y = \sigma z(part\_of(z, x) \vee part\_of(z, y)) \tag{20}$$

$$x \times y = \sigma z(part\_of(z, x) \wedge part\_of(z, y)) \tag{21}$$

$$x - y = \sigma z(part\_of(z, x) \wedge \neg overlap(z, y)) \tag{22}$$

$$\sim x = \sigma z(\neg overlap(z, x)) \tag{23}$$

$$U = \sigma z(part\_of(z, z)) \tag{24}$$

Given these basics, we can proceed to its mathematical analysis and some interesting properties, which are described in the next section.

2.2 GEM and set theory

Set theory provides structural relations to abstract mathematical entities called sets by using the *is_element_of* relation (see [19] for a brief online introduction, among many sources and books). However, its grounding in reality is debatable due to the many abstract ingredients, which mereology may overcome at least to some extent (see e.g. the introduction of [6] for arguments and §5.2 below). Since mereological theories are formulated in predicate logic (see above in §2.1), one can assess how they relate to set theory from a mathematical perspective, comprehensively assessed by Pontow and Schubert [30].

Extensions Given **GEM** with its algebraic operators as introduced in the previous section, fusion is inadequate from a mathematical and practical perspective because it disallows summing an arbitrary collection in an infinite domain. Instead of using "ϕ" as in (19), one can use second order logic and add formula (25). That is, let M be a non-empty subset of a domain D, then there exists a sum z of the entities of M in D such that for any entity w of the domain it holds true that w has an overlap with z iff there exists an element of M that overlaps with w.

$$\exists z \forall w(overlap(z, w) \leftrightarrow \exists x(x \in M \land overlap(x, w))) \tag{25}$$

// alternative characterization of fusion

The combination of (1-3), (12), and (25) is called *Closed General Extensional Mereology*, denoted with **GEM+**. Note that (25) implies (19), hence **GEM** is a subtheory of **GEM+**.

Recollecting the philosophical aversion of adding bottom, i.e. a null entity, because ontologically there does not seem to be an entity that is part of all other entities, this is undesirable from a mathematical perspective because it prevents models of such theories from getting characterised as Boolean algebras. Thus, we add a unique least element (26).

$$\exists 0 \forall z(part\_of(0, z)) \tag{26}$$

// addition of the null element (bottom)

Adding (26) requires some additions to ensure non-trivial overlap *overlap′*, non-trivial part_of *part_of′*, non-trivial proper_part_of *proper_part_of′* and non-trivial atomicity *Atom′*. That is, (5) needs an additional "$\land z \neq 0$" and so forth. As a knock-on effect, the strong supplementation (12) and two fusions ((19) and (25)) have to be redefined with their non-trivial counterparts. The addition of the null element to **GEM+** gives us *Closed General Extensional Mereology with Null*, abbreviated as $\mathbf{GEM+}^0$. Likewise, **GEM** with null becomes $\mathbf{GEM}^0$.

A third variation concerns atomicity, with atom defined as (27). A mereological theory commits to either that there are atoms (28) or entities are infinitely divisible into parts (29); see also [44] section 5, and [42] who calls them *discrete* and *continuous*, respectively[3].

$$Atom(x) \triangleq \neg \exists y \; proper\_part\_of(y, x) \tag{27}$$

// indivisable entity: an atom has no parts

$$\forall x \exists y(Atom(y) \land part\_of(y, x)) \tag{28}$$

// there are atoms

$$\forall x \exists y \; proper\_part\_of(y, x) \tag{29}$$

[3] Sowa also considers *lumpy*: some things are atoms, some are continuous. $\exists x Atom(x) \land \exists y \forall z(z \leq y) \rightarrow \exists w(w < z))$.

// entities are infinitely divisable, i.e. there are no atoms

The atomic variant adds an **A** to the abbreviation, and an atomless mereological theory has an $\bar{\mathbf{A}}$ added to the abbreviation.

Comparison Consider ZFC-set theory and recollect that a) for all sets x there exists a set y which is not an element of x, and b) there exists an injective but no surjective mapping from a set onto its power set. Also note the difference between the set theoretical level used for specification ('outer perspective') and the set theoretical level within the model ('inner perspective') at the object level; sets belonging to the latter will be prefixed with a "U-" and the U-element relation denoted with $\in^U$. Then, all U-sets are elements of the set U which itself is not a U-set, and the U-element relation $\in^U$ is a subset of $U \times U$.

The obvious 'mapping' between set theory and mereology is to map the set theoretical inclusion "$\subset$" to the *part_of* and the aim is now to try to find models of the introduced mereological theories within a given universe of sets.

A first observation is, that with an interpretation $(U, \subset^U)$ that contains the U-empty set $\emptyset$, **GEM** and **GEM+** cannot be satisfied, because they both lack the null entity. On the other hand, with the bottom as in (26) that is included in $\mathbf{GEM}^0$ and $\mathbf{GEM+}^0$, this is valid under the considered set theoretical interpretation because $\emptyset$ is the set theoretical counterpart to (26). In addition, (1-3) also translate to valid theorems of ZFC-set theory and proper parthood (4) translates to the $\subset^U$. Concerning $\mathbf{GEM+}^0$, the non-trivial version of strong supplementation can also be derived from ZFC-set theory: if a and b are U-sets and if a is not a U-subset of b then there exists a U-element c of a which is not a U-element of b, thus, the U-set $\{\mathrm{c}\}^U$ is a U-non-empty U-subset of a but is U-disjoint from b. However, taking into account the non-trivial version of the alternative fusion formula (25), then we can derive that there exists a U-set b which is not U-disjoint from any other U-set c, but such a U-set b does not exist (see [30] section 3 for details). Therefore, one can conclude that the axiom scheme with the non-trivial version of the alternative fusion formula is not true in the set theoretical interpretation. Due to the non-trivial fusion (hence also non-trivial version of the alternative fusion), the considered set theoretical interpretation is not a model of $\mathbf{GEM}^0$ and $\mathbf{GEM+}^0$ either.

Next, we consider a set theoretical interpretation that has a restriction such that it is closed under the operations of binary U-union and binary U-intersection, whose interpretation is denoted with $(U_a, \subset^{U_a})$. Following Pontow and Schubert's theorem 13 and its proof [30], $(U_a, \subset^{U_a})$ is a model of $\mathbf{AGEM}^0$ (atomic GEM with null element). Most notably, the non-trivial version of unrestricted fusion (19) is satisfied by this interpretation. It is not the case that the non-trivial alternative fusion (25) is satisfied by $(U_a, \subset^{U_a})$ if U is infinitely countable (from the outer perspective) and if the set $\{x \in U | x \in^U a\}$ is an infinitely countable (from the outer perspective) subset of U. As corollary, we get: if ZFC-set theory is consistent, there exists a model of $\mathbf{GEM}^0$ which is not a model of $\mathbf{GEM+}^0$ and there exists a model of **GEM** which is not a model of **GEM+**.

So, we still need to find a class of set theoretical structures such that also the non-trivial version of the alternative fusion can be satisfied. To get this, we need the power set $\mathcal{P}$ with an interpretation $(\mathcal{P}(a), \subset_{\mathcal{P}(a)})$. This interpretation is a model of $\mathbf{AGEM}^0$ and $\mathbf{AGEM+}^0$, because the non-trivial version of the alternative fusion is true in $(\mathcal{P}(a), \subset_{\mathcal{P}(a)})$.

Note that the converse question – if there exist models of set theory within mereological structures – can be answered in the negative, because there does not exist a greatest set in a universe of sets (see theorem 10 in [30]).

We also can assess if, and how, the models of the considered mereological theories are related to Boolean algebras. Obviously, no model of **GEM** or **GEM+** can be a Boolean algebra because they miss the null entity. But $\mathbf{GEM}^0$ and $\mathbf{GEM+}^0$ carry the structure of Boolean algebras and complete Boolean algebras, respectively. Also the converse holds, i.e. that any complete Boolean algebra is a model of $\mathbf{GEM+}^0$. Consult [30] for the theorems and proofs. Additional theorems and proofs can be found in [30] (section 5) regarding the duality between mereological theories and set theoretical, algebraical, and topological aspects, which is summarised in their theorem 34:

Theorem 34 ([30]).
(1) Any model of $\mathbf{GEM+}^0$ is isomorphic to the complete Boolean algebra of open regular sets of a Boolean topological space. In particular, the operators of binary sum and binary product are mapped to binary union and binary intersection respectively, while the general sum and general product operators are mapped to the closure of arbitrary union and the interior of arbitrary intersection, respectively.
(2) Conversely, in any topological space X the system of open regular subsets is a model of $\mathbf{GEM+}^0$ with the interior of the closure of binary union as binary sum, binary intersection as binary product, and with the interior of the closure of arbitrary union and the interior of arbitrary intersection as general sum and general product, respectively.
(3) Any model of **GEM+** is isomorphic to the complete Boolean algebra of open regular sets of a Boolean topological space where the empty set is removed from the system, and conversely, the complete Boolean algebra of open regular sets without the empty set of any topological space is a model of **GEM+**.
(4) Any model of $\mathbf{GEM}^0$ is isomorphic to a Boolean subalgebra of the complete Boolean algebra of open regular sets of a Boolean space. This subalgebra is not necessarily complete if ZFC is consistent.
(5) Any model of **GEM** is isomorphic to a Boolean subalgebra of the complete Boolean algebra of open regular sets of a Boolean space where the empty set is removed. This subalgebra is not necessarily complete if ZFC is consistent.
(6) Any model of **AGEM** is isomorphic to a Boolean subalgebra of the complete Boolean algebra of open regular sets of a Boolean space X without the empty set and with the additional property that the union of the atoms is a dense subset of X. This subalgebra is not necessarily complete if ZFC is consistent.
(7) Any model of $\bar{\mathbf{A}}\mathbf{GEM+}$ is isomorphic to the complete Boolean algebra of open regular sets of a perfect Boolean space without the empty set, and conversely, the complete Boolean algebra of open regular sets of a perfect Boolean space where the empty set is removed, is a model of $\bar{\mathbf{A}}\mathbf{GEM+}$.
(8) Any model of $\bar{\mathbf{A}}\mathbf{GEM}$ is isomorphic to a subalgebra of the complete Boolean algebra of open regular sets of a perfect Boolean space where the empty set is removed. This subalgebra is not necessarily complete if ZFC is consistent.

In short (and largely copied from [30] p135-136), there are links between mereological theories, variations on **GEM** in particular, with set theory. Theories of **GEM** and

GEM+ are not equivalent[4] (provided that ZFC is consistent). Equivalence seemed to have been tacitly assumed in literature where models of **GEM** and **AGEM** have been classified as complete Boolean algebras without the null element, which is in general only true for models of **GEM+** and **AGEM+** if ZFC is consistent. In **GEM** and **GEM+** the existence of a universal entity is stipulated by the respective fusion axioms while in set theory the existence of such an entity is contradictory, and in **GEM** and **GEM+** the existence of a least or empty entity is denied while the existence of the empty set is an ingredient of set theory. Another important difference between these theories is that in **GEM** and **GEM+** there is no straightforward analogue to the set theoretical is-element-of relation. Structures of **GEM+** and complete Boolean algebras differ only in one point: in contrary to structures of **GEM+**, in complete Boolean algebras a least element is included. Further, models of **GEM** are Boolean algebras which are not necessarily complete if ZFC is consistent. By Stone's duality theory, the models of **GEM+** can be characterized as the complete Boolean algebras of open regular sets of Boolean spaces without the empty set while models of **GEM** are isomorphic to subalgebras of these Boolean algebras without the empty set. It is remarkable that the respective isomorpisms map binary operations from the mereologies to the respective binary set theoretical operations while infinite operators are mapped to topological variants of the respective set theoretical operators.

Open issues are summarised in §6 below.

2.3 Some other extensions, variations, and gaps

The basics of mereology are considered insufficient for various reasons and have been extended to meet a wide range of divergent aims. Philosophically, the major broadening of basic mereology goes in the direction of mereotopology, i.e. combining parthood with space or location (e.g. [9] [45] and RCC8 (Region Connection Calculus)). Mereotopological relations include, among others, adjacency, partial overlap, and (non-) tangential proper part. Also, preliminary steps have been taken to somehow relate mereology to granularity (e.g. [6] [21] [27]), which range from being the only theory for granularity, to one of the relations to relate levels of granularity, to the opposite where finer-grained parts are not real parts of a whole but are somehow a collective alike the member-bunch or aggregation relation better known in conceptual modelling (see below). It is outside the scope to go into the details of these two topics. Conversely, instead of extending mereology, it can be incorporated as such in larger theoroes; for instance, the foundational ontology DOLCE [24] adheres to (contains) **GEM**.

Another direction is to look at how mereology can be mapped onto decidable fragments of first order logic with the aim to be able to use some mereological theory for computation. This will be addressed in §4.1, because at present a sub-theory has been formally mapped only to Description Logics [7], whereas mappings to other

[4] Ontologically, they differ on one point: "in **GEM** only those collections of entities may be summed up which can be characterized by an expression of the logical language, while in **GEM+** for any collection of entities a general sum is defined. From the mathematical viewpoint there is one more important difference between **GEM** and **GEM+**, namely that only the axiomatics of GEM can be expressed in the language of first order predicate logic related to the symbol set SM while GEM+ needs second order logic expressions of this language" [30] p135.

conceptual and computational languages do not necessarily adhere to mereology (see §4). The latter is in no small part due to the 'interference' of cognitive aspects and, to some extent, the 'meddling' of meronymy (part-whole relations motivated by its use in natural language) in mereology proper. Active discussions on mereology-meronomy tend to end up zooming in on two themes: (in)transitivity and types of parthood-relations (see §3). This does not imply mereology proper has no particular practical relevance for linguistics and conceptual modelling, but requires further analysis and disambiguation, which will be introduced in §3 and §4.

A third avenue is to investigate how actually the 'some arbitrary domain' of mereology and mereotopology can be used in *specific* subject domains. It is this author's impression (or bias), that attempts focus mainly on using theories of mereology for the biological and biomedical domains (e.g. [13] [39] [38] [34] [35] [40])[5]. For instance, Smith et al [40] have developed the Relationship Ontology for biomedical ontologies, which contains informal and formal definitions for several mereological and meretopological relations on both the instance level and the class level, has constraints on the relata (being endurants, perdurants, or spatial regions), are time-indexed for endurants, and the *part_of* relation between classes uses the 'all-some' construction. The latter has the following definition for endurants (taken from [40] and amended for reasons of clarity):

> **Definition (parthood between classes).** X *part_of* $Y =_{def}$ for <u>all</u> x that are instances of X, at time t, if Xxt then there is <u>some</u> y (that is an instance of Y and where X and Y are distinct) such that Yyt and x *part_of* y at t.

This means that all Xs, whenever they exist, exist as parts of Ys, according to [40]. With such a definition, one can create a plethora of other parthood relations, like temporary parthood, an X *initial_part_of* Y (every x begins to exist as part of some y), or drop the time component and let X and Y be perdurants. It leaves the door open for a more liberal definition where *some* X is part of *some* Y and where part X may outlive whole Y.

Arguably, there are 'gaps' or blank spots in mereology. How many variations, or versions, of the parthood relation are there? What about the inverse relation *has_part*? How do the parts of a whole relate to each other? Linguistic analyses, conceptual modelling languages, and domain modelling (be it ontology development or the conceptual analysis stage in software development) struggle to accommodate more part-whole aspects than mereology seems to cover for, and there are as many suggestions, workarounds and (partial) solutions on offer. This is the topic of the next sections.

3 Types of part-whole relations

Contrary to the straightforward mereological theories with the transitivity of the *part_of* relation, extensions and modifications have been proposed to a) accommodate different types of parthood relations and b) admit intransitivity in some cases of

[5] In contrast with the part-whole relation in conceptual modelling languages, which is largely focussed on the enterprise domain.

part-whole relations. See e.g. [20] [29] [46] for some discussion on the transitivity of the parthood relation. On closer inspection, it appears that in case of different types of part-whole relations, different kinds of universals are related, and, provided one makes the required distinctions, transitivity still holds (see also [46]). For instance, it is common to relate a process to its part-processes as *involved_in* to distinguish it from *part_of* between endurants (object types). Each type of part-whole relation then has to be extended with constraints on the participating object types, alike

$$involved\_in(x, y) \triangleq proper\_part\_of(x, y) \wedge Process(x) \wedge Process(y) \tag{30}$$

Other variants include relating object types spatially through the part-whole relation, denoted as *contained_in* [7] or *located_in* for relating spatial (geographical) objects. However, these labels can be deceptive, because in some cases the relata refer to the spatial *region* only, which is an abstract entity, but not the entity that occupies the region. For instance, *Book contained_in Bag* may refer to the entities themselves or to regions in space with x, y, and z coordinates. Also, what some authors consider to be 'process' needs qualification. In the DOLCE foundational ontology, process is a subtype of perdurant[6], where perdurant is generally considered to be equivalent to occurrent, which in turn is considered informally to be a synonym with process; but these different assumptions can make the understanding of a *involved_in* relation inconsistent. Specifying subtypes of parthood relations that constrain the relata requires commitment to a foundational (top-level) ontology to ensure unambiguous definitions of those parthood relations.

An important distinction exist between the mereological *part_of* relation and meronymic part-whole relations in linguistics: the latter is not necessarily transitive and may not fit within any mereological theory. For instance, *member_of*, also referred to as "member-bunch" [29], is an intransitive meronymic part-whole relation; like players are members of a rugby team, probably member of that team's club, but as player certainly not member of the rugby clubs federation. We illustrate (in-)transitivity of several mereological and meronymic part-whole relations in the following examples. The the names of the relations are extended or modified in most examples in order to indicate their 'type' other than just 'part-of'.

Example To clarify what is being part of the whole and how, we have extended the labels of the relation in most examples, such that the (in-)transitivity can be clear from the readings.

⋆ - Centimeter part of Decimeter
- Decimeter part of Meter
therefore Centimeter part of Meter
- Meter partOf SI
but *not* Centimeter part of SI, because meter is actually a *member of* the Système International d'Units.

⋆ - Vase constituted of Clay
- Clay has structural part GrainOfSand
but *not* Vase constituted of GrainOfSand

⋆ - CellMembrane structural part of Cell
- Cell contained in Blood
but *not* CellMembrane structural part of Blood

[6] Other perdurants belong to categories like event, state, achievement and accomplishment.

- Receptor structural part of CellMembrane
therefore Receptor structural part of Cell

⋆ - Employee member of Company
- Company located in Bolzano
therefore Employee located in Bolzano? but *not* Employee memberOf Bolzano

⋆ - ReceptorBindingSite regional part of Receptor
- Receptor functional part of SecondMessengerSystem
therefore ReceptorBindingSite functional part of SecondMessengerSystem?

One can extend these examples by taking into account if the relation is mandatory or not, and if their inverse relations hold. ◇

During conceptual modelling of real subject domains these considerations, and confusions, are salient aspects in the modelling exercise. For this reason, we first look into treatise by Odell [29] to clarify several aspects. This contribution is well-known in Object-Oriented conceptual modelling, hence the approach is a 'bottom up' one as opposed to the original departure of mereology. Afterwards (in §3.2), we return to taxonomies of types of parthood relation.

3.1 Part-whole relations and Odell's aggregations

It is important to first make the switch to engineering usefulness here; what goes on in engineering can at times be useful to make clarifications of theoretical approaches, and even may induce avenues for research to improve theories. In addition, sometimes things just work in practice – well, sort of and possibly sufficiently well[7] – even though purist likely will disagree. On the other hand, it is a good exercise for mereologists to analyse relations that are informally grouped under the heading of part-whole relations. Odell's suggestions for part-whole relations are summarised in Table 1, and discussed in the remainder of this section.

Table 1: Odell's types of part-of relations.

Type of part-of	**Explanation**
component – integral object	Discrete type of part-whole relation with atoms
material – object	Constitution of objects
portion – object	a) some amount of matter is part of the whole, and b) scale-based partonomic relations
place – area	Where part-place cannot be separated from the whole-area
member – bunch	Whole bunch is generally denoted with a collective noun and its members can change over time
member – partnership	Like member-bunch, but changing a member does destroy the whole

[7] Permitting a mild digression: that it 'sort of' works and possibly sufficiently does not mean mereology is implemented properly (see also §4), but some argue that it is a so-called 80-20 situation, where the relatively easy bit of parthood relations have passed the revue, but to do it well and address the remaining 20 costs too much. Conversely, it may be that only 20% is addressed and the 80% still has to be uncovered and implemented but that conceptual modelers are not aware of it and therefore do not realise what they are missing.

The first type of aggregation Odell identifies within the context of proposing them for inclusion in the class diagrams of the Unified Modeling Language (UML), is component-integral object, which corresponds to the (structural) *part_of* relation. The second one, "material-object" is ontologically *not* a mereological relation, because it deals with *constitution* of objects, like a vase is constituted of clay [24].

Third, portion-object is ambiguous, where Odell gives examples like "a slice of bread is a portion of a loaf of bread", or meter part of a kilometer [29] (and so forth for other arbitrary scales). Ontologically, we have two separate cases here: the first deals with a relation that some amount of matter is part of the whole, like a sip of wine is (or was) part of the wine in my glass of wine. In how many sips of wine can the wine in my glass be 'partitioned'? These sips are *portions*, but *not* parts. These type of objects are commonly indicated with mass nouns as opposed to count nouns, i.e. amounts of matter in DOLCE [24] as one does not refer to 1 wine, 2 wine etc, but one *sip* of wine, two *portions* of mashed potatoes, three *servings* of pudding and so forth. One caution to observe, is that an entity referred to with a mass noun does not imply that it is ontologically an example of portion-object. Note also that this is different from the sorites paradox [41], also known as the heap problem, which has to do with vagueness, not granularity or mereology. The other issue with portion-object, is that of scale-based partonomic relations, which is distinct from the amount of matter and whole entity that is categorised as of the same type. With measurements, one actually *can* count; moreover, it is an essential feature one can determine in advance what the parts are and how many of them there are (like 10dm go in 1m). There is not some fuzzy *Day* with a fiat boundary, but it has 24 hours as its parts[8]. That Odell adds homeomericity with "similarity between a portion and its whole" as a characterising property does not solve which ontological meaning should be given to the portion-object type of aggregation, but other examples Odell presents suggest the first interpretation.

Fourth, there is the "place-area" type of aggregation where the part-place *object1* occupies cannot be separated from the whole-area *object2*, where *object1* is physically smaller than, and part of, *object2*. One can refine this aggregation to be one of *containment* or *location*. With *contained_in* and *located_in*, we make the minor distinction between 3D spatial containment, like mitochondrion in cell or toolbox in the trunk of a car versus those that can be 'mapped' 2D, like oasis-desert or the region Alto Adige in Italy (province-country).

The fifth type is "member-bunch", which, as mentioned above, is not a mereological relation because it is not transitive. The sixth aggregation type "member-partnership" puts another constraint on the fifth type, such that changing a member *does* destroy the whole. Although not mentioned by Odell, this association (relation) requires to have not only a cardinality constraint of 2 for e.g. a marriage and Dick und Doof, but also constraints on which instances are allowed to participate in the partnership (e.g. only Stan Laurel and Oliver Hardy can make the partnership of Dick und Doof). However, one can also mode this as *participates_in*, where a marriage is a certain

[8] A digression into assessing the ontological nature of each type of (SI) measurement scale is beyond the current scope.

perdurant in which two persons participate. Note that the member-partnership put a constraint on specific individuals participating in the relation, which is an aspect not addressed in mereology.

Summarising, this leaves component-integral object (proper part of), place-area (located in), option b of portion-object, and member-bunch as types of part-whole relations. The first two can be categorised as simplified descriptions of mereology and mereotopology respectively; simplified, because it it not clear regarding constraints that, for instance, the component *must* be part of the integral object and if it can exist as component of that (type of) integral object only (it is ignorant about existential, mandatory and contingent parts). Member-bunch is semantically a different type of part-whoel relation.

3.2 A taxonomy of part-whole relations

Odell's proposal for part-whole relations consists of a list of relations, but efforts have gone into constructing a taxonomy of part-whole relations. The first proposal, motivated by linguistic use of 'part', i.e. meronymy, was made by Winston, Chaffin and Herrmann [48]. Several articles deal with analysing the WCH taxonomy and modelling considerations (e.g. [1] [11] [12] [29]). For instance, Gerstl and Pribbenow [11] prefer a "common-sense theory of part-whole relations", motivated by differences *in* compositional structure of the whole compared to *external to* the whole and allows for "different views on the entities" [11](p888). Gerstl and Pribbenow reduce the six types of part-whole relations of WCH into three: component-complex, element-collection, and quantity-mass. This has been improved upon by Guizzardi [12], who distinguishes "three types of conceptual parthood", being *sub_quantity_of*, *sub_collection_of*, and *member_of*. The *sub_collection_of* is a set-subset relation, and not strictly a mereological part-of; on the other hand, this aspect can be satisfactorily addressed with the previously discussed mathematical analysis in §2.2. Guizzardi also offers *criteria* for each as opposed to example-based (ch5 and ch7 of [12]). In addition, he extends the investigation with Vieu and Aurnague's linguistically motivated functional parthood relations [47], which have ontological issues of their own (the notion of 'function' that is).[9]

To clarify these distinctions, Keet [22] developed a simple taxonomy of part-whole relations, which is depicted in Fig.2. It has a major distinction between mereological and meronymic part-of; the major reasons why this ontological distinction exist has to do with transitivity of the parthood relation, which was illustrated in the example above: mereological part-of relations are transitive, meronymic not necessarily. The member-bunch, sometimes *collection* and subsumed by the liberal term *aggregation*, belongs to the meronymic branch and is labelled with *member_of*. *sub_collection_of* is not included because of its set-theoretical emphasis; if added, then it is subsumed by *Meronymic relation*. What the figure does not show is that the 'spatial' is not in the sense of abstract region. Attempts to integrate this taxonomy with DOLCE

[9] Guizzardi also makes explicit the distinctions between essential parts (existential dependence), mandatory parts (generic constant dependence), and contingent parts. We return to this in the UML section (§4.2) because it does not affect a top-level taxonomy of types of part-whole relations.

foundational ontology (as tested with its OWL version in Protégé) would fail the consistency check & classify taxonomy if the automated reasoner would have the capability to reason over a hierarchy of relations (called RBox in Description Logics): functional parthood cannot be properly represented as differentiating from *s_part_of*, and although *contained_in* does have its relata (domain and range) constrained to physical object, *located_in* relates regions, which are in another branch in the DOLCE ontology and would be inconsistent unless the *spatial_part_of* has *PT*s as its relata (which in turn would make it equivalent to the *Mereological_part_of*). Alternatively, one may prefer to create multiple inheritance such that *contained_in* is subsumed by both *s_part_of* and *spatial_part_of*. Or take another foundational ontology (cf. SUMO, BFO, OCHRE, GOL) that will suit this taxonomy. In short, although this version is more comprehensive and makes several ontologically justified distinctions, it is still up for debate, improvement, and possible extensions.

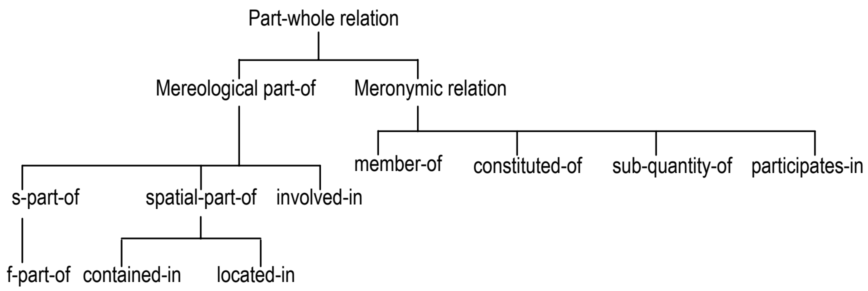

Fig. 2: Taxonomy of basic mereological and meronymic part-of relations. s-part-of = structural part-of; f-part-of = functional part-of. See text for details on the relata of the relations.

Reflecting on the plethora of types of part-whole relations and the 'clean' mereological theories, mereology can help assessing what is and what is not a true parthood relation and aids structuring the relations, but from the moment one starts looking at the relata, i.e. the kind of entities that stand in a part-whole relation to each other, other aspects have to be taken into account, like ontological categories of the types of entities and the (linguistic) use of parthood.

4 Part-whole relations in conceptual modelling languages

Considering the variations in mereological and meronymic part-whole relations, it is useful to look into how the various conceptual data modelling languages treat such relations, which varies for each language and user community. It is also known under the term *aggregation* relation, although aggregation differs from part-whole relations. Subsets can be 'aggregated' into larger sets, but the set-subset semantics is not the same as the formal semantics of mereology, as we have seen in §2.2, and the aggregate doe snot necessarily correspond to an entity in reality. Further, at times the intention is to model a mereological *part_of* instead of just grouping entities, but the language does not permit it fully or only with ambiguous semantics. In this section we take a closer look at four languages and how they fare regarding the part-whole relation: Description Logics (in §4.1), UML class diagrams (§4.2) and EER and ORM (§4.3).

4.1 Description Logics

Description Logics is more often used as a knowledge representation language than as conceptual modelling language, and if used for conceptual modelling it is normally used 'in the background' and restricted to $\mathcal{DLR}$ [10] [5]. As neither UML nor ER nor ORM, which are mapped to $\mathcal{DLR}$, implement part-whole relations properly (elaborated on below), it may not matter that $\mathcal{DLR}$ does not have a comprehensive treatment of part-whole relations – at present. It is, however, being investigated.

Some options to represent parthood relations in some DL language Besides inadequately defining a `part_of` or `has_part` role in a DL, we can add a *has_part* role as $\succeq$, model it as the transitive closure of a parthood relation (31) and define e.g. `Car` as having wheels that in turn have tires [1] (32), such that it follows that `Car` $\sqsubseteq \exists \succeq$.`Tire`.

$$\succeq \doteq (\texttt{primitive-part})* \tag{31}$$

$$\texttt{Car} \doteq \exists \succeq .(\texttt{Wheel} \sqcap \exists \succeq .\texttt{Tire}) \tag{32}$$

However, transitive closure in $\mathcal{ALC}$ is EXPTIME-complete. Alternatively, one can define direct parthood $\prec_d$, but this should verify the immediate inferior, which makes the language undecidable [2]. Schulz et al [34] have developed a workaround for $\mathcal{ALC}$ to get to reason with transitive parthood relations. They remodel transitive part-of relations as *is_a* hierarchies using SEP triplets. The three core items are the **s**tructure-concept node that subsumes one (anatomical) entity, called **e**ntity node, and the parts of that entity (the **p**-node). The *is_a* hierarchy is then built up by relating the P-node of a whole concept D to the S-node of the of the part C, where in turn the P-node of C is linked to the S-node of C's part; see Fig.3. More formally, the definition of the whole D is (33), by which one can derive its anatomical proper part C as (34).

$$D_P \doteq D_S \sqcap \neg D_E \sqcap \exists a\text{-}pp.D_E \tag{33}$$

$$C_E \sqsubseteq \exists a\text{-}pp.D_E \tag{34}$$

Around the same time, Sattler [33] showed that with some extensions to $\mathcal{ALC}$, it is possible to include more aspects of the parthood relation. These are: transitive roles (that is, permit $R_+ \subseteq R$), inverse roles to have both part-of and has-part, role hierarchies to include subtypes of the parthood relation, and number restrictions to model the amount of parts that go in the whole. This brings us to $\mathcal{SHIQ}$.

The latest, and most comprehensive attempt to represent parthood relations in DL is by Bittner and Donnelly [7], who approach the problem starting from the FOL characterisation and subsequently limiting its comprehensiveness and complexity to fit it into a DL language, although it is unclear if their $\mathcal{L}^{\sim Id \sqcup}$ is decidable. In their theory DL-PCC, several constraints and definitions cannot be represented. These are:

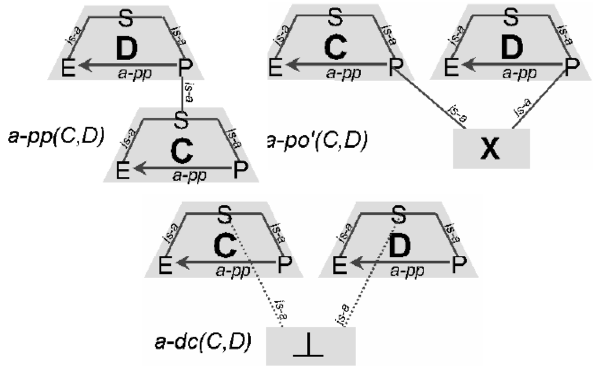

Fig. 3: Figure 5: SEP Triplet Encoding for Anatomical Proper Part (*a-pp*), Anatomically Disconnected (*a-dc*), and Anatomical Partial Overlap (*a-po'*). (Source: after [34])

impossibility to state that *component_of* (CP), *proper_part_of* (PP) and *contained_in* (CT) are irreflexive and asymmetric, and it is missing a discreteness axiom for CP or CT or a density axiom for PP (see [7] for details and discussion). Noteworthy is that they include transitivity of the characterised parthood relations as standard in mereology, but thereby do not have the option to state also that e.g. a *directly_contained_in* relation is *in*transitive.

Other considerations Artale et al [1] [2] has placed the requirement for adequately representing the part-whole relation in a wider context, where some outstanding issues of 10 years ago are still in need of a solution. For instance, (non)distributivity of certain roles[10] [2]. A main issue revolved around the role composition operator and transitive closure of 'direct' parthood roles at the TBox level [1], i.e., $\exists W.\exists W.C \sqsubseteq \exists W.C$), where the computation does derive that if part x that is a direct part of y that in turn is a direct part of the whole w, then x is a part of w but not that x is a *direct* part of w. This, however, assumes one needs a relation defined a *direct_part_of* as pursued by Sattler [32], which is neither necessary from an engineering perspective nor desirable from an ontological (mereological) point of view. Alternatively, one can use the role composition operator to denote transitivity $R \circ R \sqsubseteq R$. Role composition has been developed further up to role inclusion and concatenation in $\mathcal{SROIQ}$ [17]. However, note the two existential quantifications and that this is at the TBox level: the former is quite distinct from the all-some definitions in the Relationship Ontology (cf. §2.3) and the latter was already briefly mentioned in [2] section 3 with corresponding unresolved issues regarding differences between TBox and ABox reasoning with the parthood relation.

Another aspect of part-whole relations is the absence of 'horizontal' relations between the parts, already noted by Artale et al in 1996 [1], which is a problem for DL

[10] Downward distributive: there are properties of the whole that the parts inherit; upward distributive: the whole inherits properties form its parts. Alternatively, they are called property inheritance through parts and property refinement through parts.

representations, but less problematic for conceptual models for databases. DL adheres to the open world assumption, thus if we have in a DL TBox a statement alike

$\mathtt{C} \sqsubseteq \exists\, \mathtt{has\_part.D} \sqcap \exists\, \mathtt{has\_part.E}$

then it may be that C has more parts than only D and E and the composite C may not be fully defined. No DL language deals with an additional axiom that states that C is *composed* of, the mereological sum of, D and E *only*. In contradistinction, conceptual models do adhere to a closed world assumption, thereby making C uniquely composed of at least one (instance of) D and at least one (instance of) E (depending on the constructors in the language, see below). For both, however, one needs additional relations (roles) to declare how the parts themselves relate to each other. For instance, let C be Book, D be BookCover, and E is Page, then an additional horizontal role has to be declared alike $\mathtt{BookCover} \sqcap \exists\, \mathtt{encloses.Page}$. This is not impossible, but its consequences with respect to the parthood relation are not known. The status of differentiating between e.g. essential and mandatory part (see §4.2) is unclear, idem ditto the distributivity.

Last, Bittner and Donnelly [7] reach an interesting conclusion with respect to feasibility of representing parthood in DLs, which merits further investigation: " DLs are best used as reasoning tools for specific tasks in specific domains... [but] are not appropriate for formulating complex interrelations between relations. Thus we need to understand a computational ontology as consisting of two complementary components: (1) a DL based ontology that enables automatic reasoning and constrains meaning as much as possible and (2) a first order ontology that serves as meta-data and makes explicit properties of relations that cannot be expressed in computationally efficient description logics. The first order theory then can be used by a human being to decide whether or not the DL-ontology in question is applicable to her domain. Moreover, meta-data can also be used to write special-purpose programs that phrase knowledge bases and enforce the usage of relations in accordance to the meta-data." [7]. Given that reasoning over an ontology, knowledge base, or conceptual model is desirable, transitivity is of greater importance than observing direct parthood, therefore a restricted theory of parthood relations like provided by Bittner and Donnelly may be more useful than direct parthood and SEP triples, although a thorough comparison is yet to be conducted.

4.2 Aggregation in UML

UML specification UML [28] offers aggregation in two modes: composite and shared aggregation. Composite aggregation, denoted with a filled diamond on the whole-side of the association (see Fig.4), is defined as

> a strong form of aggregation that requires a part instance be included in at most one composite at a time. If a composite is deleted, all of its parts are normally deleted with it. Note that a part can (where allowed) be removed from a composite before the composite is deleted, and thus not be deleted as part of the composite. Compositions define transitive asymmetric relationships – their links form a directed, acyclic graph. [28]

The composite object is responsible for the existence and storage of the parts [28], where this 'implementation behaviour' of creation/destruction is an implicit *ontological commitment at the conceptual level*: the parts are existentially dependent on the whole (which implies mandatoryness), and not that when a whole is destroyed its parts can exist independently and become part of another whole. On the other hand, the whole is neither existentially dependent on its part nor has the part mandatorily. In addition, only binary associations can be aggregations [28], which is peculiar from an ontological perspective as this may (or may not) suggest that the aggregate/composite can have only one type of part. For instance, we have a composite aggregation between *Lysosome* and the whole *Cell*, then the UML composite aggregation says not only that lysosomes are existentially dependent on the cell they are part of and that they can be part of one cell only, but also that the cell's parts are lysosomes *only*, which is biologically incorrect. More general and precise, let A be the whole with parts B, C, and D in a UML class diagram as in Fig.4-A, then each part is associated to the whole through a separate binary composite aggregation, as if A is a whole where its instance a is made up of a collection of instances of type B, and/or made up of a collection of instances of type C and/or D, making A a different type of entity for each aggregation of its parts, which cannot be the intention of the representation because that does not have a correspondence with the portion of the real world it is supposed to represent. What needs to be represented, and supported in the language, is that instances of B, C, and D *together* make up the instance of the whole entity A, as in Fig.4-B, and prevent a modeller to create something like Fig.4-A. This difference is

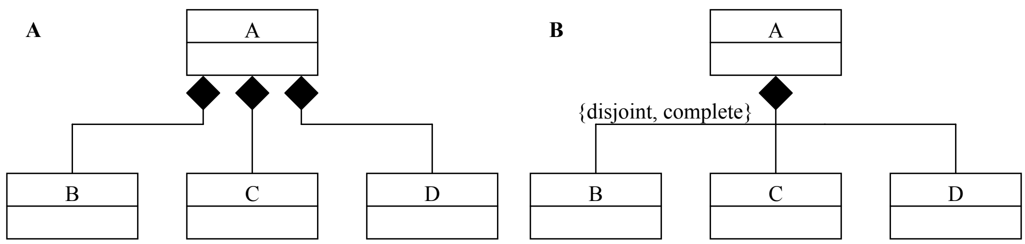

Fig. 4: A: Ontologically ambiguous UML composite aggregation as separate binary associations; B: the composite A is composed of parts B, C, and D.

not mentioned in the UML specification, and one is left to assume it probably is a "semantic variation point" which of the two readings should be used (like the precise lifecycle semantics of aggregation is a semantic variation point too) [28]. Likewise for shared aggregation, which is denoted with an open diamond on the whole-side of the aggregation association, that has it that "precise semantics ... varies by application area and modeler" [28], and presumably can be used for any of the part-whole relations described in Fig.2, or by [20] [25] [29] [48] etc. Unlike composite aggregation, shared aggregation has no constraint on multiplicity with respect to the whole it is part of. Thus, the part may be *directly shared* by more than one whole at the same time. The latter can be interpreted in two distinct ways. Let $W1$ and $W2$ be disjoint classes on the aggregation side, and P the part class that is aggregated, then i) an instance p_i is part of instance $w1_1$ and possibly also part of instance $w1_2$ (or vv.) and possibly more instances of $W1$, and/or ii) an instance p_i is at least part of instance $w1_1$ and possibly also part of instance $w2_1$ (or vv.). An example of the former is that all instances of BSc curricula at the FUB must have as part a course in ethics, and of the latter, that a particular seminar may be part of both the LCT Colloquia and of the KBDB course. As exercise, try to come up with an example involving physical objects, and how to represent the different semantics in UML.

Overall, the ambiguous specification and modelling freedom in UML does not enable making implicit semantics explicit in the conceptual model, and rather fosters creation of unintended models. This has been observed by several researchers, who have proposed a range of extensions to UML class diagrams, which is discussed in the next section.

Formalizations of aggregation in UML class diagrams UML does not have a formal semantics, which demands from the researchers who propose extensions to also give the formal semantics. Whereas Barbier et al [4] maps it into the, less than optimal, Object Constraint Language, Motschnig-Pitrik and Kaasbøll [25] and Guizzardi [12] present a First Order Logic formalization and corresponding 'dressing up' of the graphical notation with several new icons and labels.

This is in stark contrast with the minimalist approach taken by Berardi et al [5], who do not take into account these extensions and limit the formalisation to shared aggregation only (no formalisation is provided to account for additional constraints and composite aggregation). This avoidance is partially due to the ambiguous semantics of aggregation in UML and because adequately representing parthood in DL is not an easy task, as was discussed in §4.1. In Berardi et al's [5] formalisation of shared aggregation in UML class diagrams, we have (35), where G is a binary predicate (for the aggregation, or 'part_of') and C a concept.

$$\forall x, y(G(x, y) \rightarrow C_1(x) \wedge C_2(y) \qquad (35)$$

Here we present a few formal definitions regarding parthood relations, which are in *addition* to the axiomatization in mereology and perceived to be necessary for conceptual modelling of mereological and meronymic relations, and for which there is also a correspondence in the UML extensions and corresponding graphical modelling language. Guizzardi adds, among others, the notion of essential part EP, defined as ([12]: p165):

> **Definition 5.11 (essential part)**: An individual x is an essential part of another individual y iff, y is existentially dependent on x and x is, necessarily, a part of y: $EP(x, y) =_{def} ed(y, x) \wedge \Box(x \leq y)$. This is equivalent to stating that $EP(x, y) =_{def} \Box(\epsilon(y) \rightarrow \epsilon(x)) \wedge \Box(x \leq y)$, which is, in turn, equivalent to $EP(x, y) =_{def} \Box(\epsilon(y) \rightarrow \epsilon(x) \wedge (x \leq y))$. We adopt here the mereological continuism defended by (Simons, 1987), which states that the part-whole relation should only be considered to hold among existents, i.e., $\forall x, y(x \leq y) \rightarrow \epsilon(x) \wedge \epsilon(y)$. As a consequence, we can have this definition in its final simplification
>
> (47). $EP(x, y) =_{def} \Box(\epsilon(y) \rightarrow (x \leq y))$

where ϵ denotes existence, $\leq$ for a partial order, and the $\Box$ necessity. The 'weaker' version is mandatory parthood MP, defined as ([12]: p167):

Definition 5.13 (mandatory part): An individual x is a mandatory part of another individual y iff, y is generically dependent of an universal U that x instantiates, and y has, necessarily, as a part an instance of U:
(49). $MP(U, y) =_{def} \Box(\epsilon(y) \rightarrow (\exists U, x)(x < y))$.

To compare the mandatory part with the relation into the other direction, i.e. a mandatory whole MW, we have ([12]: p170):

Definition 5.16 (mandatory whole): An individual y is a mandatory whole for another individual x iff, x is generically dependent on a universal U that y instantiates, and x is, necessarily, part of an individual instantiating U:
(52). $MW(U, x) =_{def} \Box(\epsilon(x) \rightarrow (\exists U, y)(x < y)))$.

Last, we add an example of 'dressing up' UML class diagrams, which is depicted in Fig.5, demonstrating the proposed representation for the *sub_quantity_of* relation with an additional symbol, OCL constraint, and stereotypes. We can compare this with

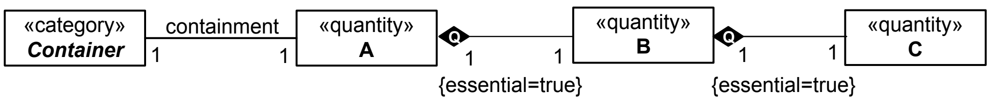

Fig. 5: Part-whole relations among quantities. Essential parts are indicated with `essential = true`, which implies a composite aggregation (filled diamond), which is of the type "Q" for quantities. The stereotypes ("≪≫") add further constraints to the permitted types of classes. (Source: [12])

other formalisations, approaches and emphases. For instance, Motschnig-Pitrik and Kaasbøll focus on, among other things, *gradations* of exclusiveness between part and whole. This corresponds partially to Guizzardi's mandatoryness and (in)separability of the part from the whole, as can be observed from one of the definitions, like for total exclusiveness ([25]: p785):

Total exclusiveness. A part-of reference is *totally exclusive* if there exists exactly one immediate part-of link from a part-type P to a whole-type W and, for each instance p of P, there exists at most one instance w of W such that p part-of w. More formally, let:
p_k instance-of $P, k \in [1..n]$,
w_i instance-of $W, i \in [1..n'], w_{i'}$ instance-of $W, i' \in [1..n]$,
wx_j instance-of $WX, j \in [1..n'']$
then
P totally-exclusive part-of $W \Leftrightarrow \forall WX$
$((P$ part-of $W \wedge P$ part-of $WX) \Rightarrow (W = WX \vee W$ part-of $WX)) \wedge$
$((p_k$ part-of $w_i \wedge p_k$ part-of $w_{i'})) \Rightarrow (i = i'))\wedge$
$((p_k$ part-of $w_i \wedge p_k$ part-of $wx_j) \Rightarrow (w_i = wx_j \vee w_i$ part-of $wx_j))$

Then one can add further gradations, such as arbitrary sharing, interclass exclusiveness, intraclass exclusiveness, and selective exclusiveness, each with their respective label attached to the association, like *SelExcl* and so forth [25]. Barbier et al [4] formulate the various add-ons as a "context" in OCL and add a meta-model fragment for whole-part relations where the attribute `aggregation` is removed from the *AssociationEnd* meta-class, *Whole* added as a subclass of *Relationship* and has two disjoint subclasses *Aggregation* and *Composite*.

As the time of writing this document, none of the proposed extensions have made it into the UML specification. Considering the amount of effort put in investigating and proposing extensions to UML over the past 15 years, it will be useful to look into *why* the treatment of part-of remains minimal and ambiguous in the UML specification.

4.3 Part-whole relations in (E)ER and ORM

It may be clear from the previous section that part-whole relations in UML class diagrams can have poorly defined semantics, but what about other conceptual modelling languages? Entity-Relationship (ER) does not have a separate constructor for the part-whole relations, despite the occasional [36] request. Neither does the Object-Role Modelling (ORM) language have a separate constructor for parthood relation[11]. Are they better off than UML? What, if any, can already be represented from part-whole relations with ER or ORM? Here, we assess ORM.

Recollecting the UML specification that inserts design and implementation considerations for composite aggregation (that the parts are *existentially dependent* on the whole, and not that when the whole is destroyed the parts can have their own life), more is going on. A shared aggregation, such as the *member_of* relation, between *Team* and *Player* in Fig.6, does not necessarily have the parts deleted as the players may play in more than one team, but only have removed the particular team and the association between that destroyed team and its players. In contrast, a particular plasmid exists in one cell only and is a part of that cell[12], hence has to be modelled as a composite aggregate. However, destroying the cell leaves the plasmid free to go to another cell instead of being destroyed as well. Here there is a difference between UML and ORM intended semantics: with composite aggregation in UML, part x *cannot* exist without that whole y, but ORM semantics of the suggested mapping [16] says that 'if there is a relation between part x and whole y, then x must participate exactly once'. Put differently, x may become part of some other whole y' after y ceases to exist, as long as there is *some* whole of type Y it is part of, but not necessarily *the same* whole. Hence, in contrast with UML, in ORM there is no strict existential dependency of the part on the whole. Fig.6-B shows an example, where the *Club-Team* fact has its first order logic representation as (36-39) and *Team-Person* as (40-41). It does not specify

[11] Other languages, such as RDF, XML, and the DL-based OWL, are in the very early stages (too). See e.g. [26] [49] (and maybe [18]).

[12] It is outside the scope to provide an analysis if this parthood relation is a structural parthood or of a contained-in nature.

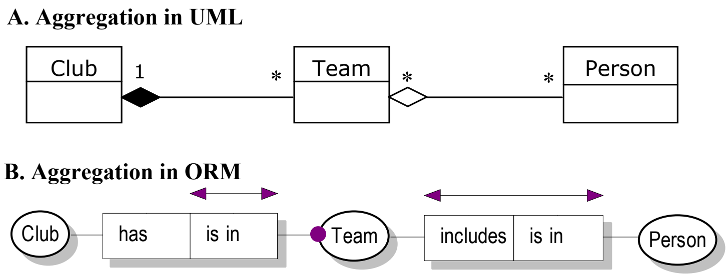

Fig. 6: Graphical representation of "aggregation" in UML and ORM. (Source: [15])

that it is impossible for x to exist independently. Thus, the plasmid-cell example can be correctly represented with ORM but not UML.

$$\forall x, y, z((isIn(x, y) \wedge isIn(x, z)) \rightarrow y = z) \tag{36}$$

$$\forall x, y(isIn(x, y) \rightarrow Team(x) \wedge Club(y)) \tag{37}$$

$$\forall x(Team(x) \rightarrow \exists y(isIn(x, y))) \tag{38}$$

$$\forall x_1, x_2(isIn(x_1, x_2) \equiv has(x_2, x_1)) \tag{39}$$

$$\forall x, y(isIn(x, y) \rightarrow Person(x) \wedge Team(y)) \tag{40}$$

$$\forall x_1, x_2(isIn(x_1, x_2) \equiv includes(x_2, x_1)) \tag{41}$$

Compared to more and less comprehensive formalizations and extensions for aggregation in UML [4] [12] [25] [5], for ORM, richer representations of the semantics are possible already even without dressing up the ORM diagram with icons and labels (see [22], who also provides several guidelines to ease selecting the appropriate part-whole relation and its mandatory and uniqueness contraints).

Last, transitivity is also referred to as derived aggregation association [16], although not specifically mentioned as such in the UML specification. Both UML and ORM allow representation of derived (aggregation) associations/roles, which, considering the freedom in representing mereological relations, may be a method to add the semantics of transitivity of the aggregation – assuming that when no derived aggregation association occurs then the aggregation is intransitive. However, if we apply ontological rigour to a model, i.e. domain knowledge is properly represented, the different types of aggregation are appropriately kept separate and each relation is transitive by default and usable for inferencing without the need to add 'workarounds' to the model with derived aggregations.

Reflecting briefly on the treatment of part-whole relations in knowledge representation and conceptual modelling languages, there seems to be little left (or: transferred from) mereology and multiple other issues have come to the surface, such as mandatory parts, essential parts, mandatory wholes, degrees of shareability, and ambiguous specifications. There are similar problems across the spectrum of languages, with multiple alternative solutions as is customary in computer science and engineering. Identifying the similar problems more generally might lead toward new developments in mereology and thereby contribute to a coherent treatment of part-whole relations across the spectrum. In the next paragraph we describe several requirements.

5 Arguments for representing and implementing part-whole relations

It may be clear from the previous sections that part-whole relations in UML class diagrams can have poorly defined semantics, DL struggles with its proper representation, ER and ORM do not treat it as a 'first-class citizen' either, mereology does not address all part-whole aspects relevant for conceptual modelling, and the notion of types of part-whole relations is not yet resolved fully. In this light, it is not surprising why the usefulness of including a separate constructor for the parthood relation in conceptual modelling languages is called into question. In this section we look at arguments why, and why not, it should be included. One might wonder why we did not start with justifications, but after treating the philosophical, mathematical, and conceptual modelling aspects, we can better compare the arguments put forward, i.e. forming an informed standpoint as opposed to having an opinion based on intuition or single-perspective focus.

5.1 Requirements

Considering the differences in definitions, implicit and explicit representations of the part-whole relations in knowledge representation languages, an overall structure and implementation would have to meet several requirements such that:

i. a 'minimal amount' of mereological *part_of* relations can be unambiguously identified and are useful for conceptual modelling, including taking into account the relata of the relation for types of part-whole relations;
ii. enable the modeler to distinguish between those types of parthood relations as well as identify the non-mereological meronymic relations;
iii. provide a subset of combinations of uniqueness and mandatoryness constraints applicable to the relation, such that they are ontologically correct/possible;
iv. clarify and accommodate for other, so-called secondary, properties of part-whole relations, such as existential dependence, inseparability, functional dependence, and completeness;
v. ensure the inverse, has-part, relation is properly modelled as well;
vi. transitivity of parthood relations is enabled where applicable and prohibited for non-transitive parthood-like relations;
vii. address the possibilities and consequences of horizontal interrelations between the parts of a whole;
viii. ensure the representation is such that one can distinguish between parthood relations of a class (or its instances) and other generic properties (/relations/roles/associations), i.e. to make part-whole and whole-part relations first-class citizens;

ix. the constructors do not only serve as 'paper exercise' but also have computational support;
x. that there is an underlying unifying paradigm that relates the conceptual modelling language specific constructors (if possible).

At the time of writing, we are far away from meeting these requirements, and even if this is realised, it is unclear if modellers and software analysts will use it. Point i can be met with the simple taxonomic structure as presented in Fig.2, and point ii with a decision procedure alike shown in figure 3 of [22]. Secondary properties, inverse relations, transitivity, horizontal relations, and distinguishing between part-of relations and other properties is only to a very limited extent possible in some languages (extensions of UML) and then either mostly implicit or with a myriad of additional labels. Full computational support likely is not possible due to undecidability, which hampers efforts achieving points ix and x; hence, requires a narrowing down to the decidable fragment.

5.2 Discussion

DL, ER and ORM deal with part-whole relations simply by naming the relation or fact type as *part_of* and *has_part*, or some similar label, and add appropriate uniqueness and mandatory constraints. On the surface, this may seem an advantage as it gives more freedom during the conceptual modelling exercise, e.g. to name a relation *contained_in*. However, if we omit the mereological part-whole construct, what we are actually modelling is how *sets* of objects relate to each other, but not how *parts* make up the whole (aggregation in UML, albeit poorly defined, does claim that the parts make up the whole). Recollecting the relation between mereology and set theory (§2.2), where mapping is possible under certain constraints, this does not do away with the ontological, philosophical, issues. This is more fundamentally described by Sowa [42] (among others), and particularly relevant if one indeed intends to model a portion of reality. First,

- *A mereological aggregate of multiple physical entities is a physical entity.* An aggregate of abstract entities is also abstract. But in the usual versions of mereology, an aggregate of a physical entity like the cat Yoyo with an abstract entity like the number 7 would be meaningless or undefined.
- In set theory, however, the axioms impose no constraints on the types of the members. A cat and a number could be both members of the set $\{Yojo, 7\}$ because it is a new entity, whose type does not depend on Yojo or 7. A reasonable interpretation is that sets are abstractions, *independent of the nature of their members*, which may be physical, abstract, or mixed. [42] (italics added)

One can argue that no conceptual modeller will define nonsensical sets such as $\{Yojo, 7\}$ anyway and that the language should be as minimal as possible instead of expressive (or restrictive) for the sake of making it foolproof. But there is an important difference between, say, being a set of molecules that make up a whole body versus that molecules are parts of your body. First, elements of a set have no additional structure among the elements (even though your molecules are in a particular configuration), but parts of a whole do have this *and* that can be represented. Second, if the elements of a set change, then so does the extension of the whole, even though the identity of a whole may remain. For instance, you get sunburned, which changes healthy DNA into DamagedDNA; the new set with instances of DamagedDNA molecules creates a new type of 'set-based whole' by virtue of its representation, but it is not the case that a change in representation results in a different person. In addition, identifying an additional part of a person does not imply a change of the whole. Third, set-based mereology requires an Urelement [6] [21], but mereology does not. In several domains, most notably the biological domain, this can lead to some undesirable ontological consequences (see [21] for examples). Fourth, elaborating on Sowa, if the extension of a set with parts is properly grounded, it represents physical objects, and therefore so should the mereological whole; this is implied with mereology, but not guaranteed with aggregating sets. Thus, granting parthood relations a status of first-class citizen allows one to represent reality more accurately.

A counter-argument concerns ease of representation versus accuracy of representation with respect to reality. Some awkward constructs like the SEP triples do not represent reality accurately and has its own system to transform parthood relations into taxonomic subsumption, but it is a useful workaround to get the software system to deal with parthood relations. Also, the Semantic Web language for ontologies OWL is in fact stored as an XML serialization, but people do not have cried foul that XML does not even deal properly with relations in general, let alone parthood relations. Likewise, ontologies containing partonomies like the Foundational Model of Anatomy [31] are stored in set-theory-based relational databases. Ontological correctness, usable computational support for, and usability of the parthood relation are not by definition always conflicting goals on all points, but there is a trade-off. For instance, the additional expressivity upon enabling proper representation of part-whole relations comes at a cost of greater difficulty during the modelling stage, although this can be ameliorated by providing guidance through Q&A sessions, decision diagrams, and drop-down boxes to facilitate selecting the appropriate relation and constraints. Likewise, mereological theories can be mapped only with limitations onto set theory.

More importantly, proper inclusion of the part-whole (and whole-part) relation can allow correct reasoning over a conceptual model – for what remains within the decidable fragment. This may be of no interest during development of a small conceptual model, but deriving implied relations, derived relations (transitivity), and satisfiability can aid correcting large conceptual models and thereby diminishing debugging time and resources during the software testing stage. Generally, advantages to include different parthood relations are a.o. automated model verification, transitivity (derived relations), semi-automated abstraction operations, and enforcing good modelling practices. On the other hand, specifying everything into the finest detail may be too restrictive, results in cluttered diagrams, is confusing to model, and costs additional resources to include in CASE tools. Incrementally adding further constraints like essential part, the whole-part relation, and inter-part relations, enable the conceptual modeller to gradually develop models that are closer to the real-world semantics and thereby improve quality of the software. When used more widely, it can be useful to add extensions to the knowledge representation languages, e.g. as a separately loadable modules in CASE tools for those modellers who need it, analogous to the Description Logics approach with a family of more and less expressive knowledge representation languages. A relative 'shortcut' toward implementation is to consider the pervasive DLs that not only provide the basis for the Web Ontology Language OWL, but a variant ($\mathcal{DLR}$ [8]) serves as underlying unifying language for conceptual modelling languages UML, ER, and ORM and already enjoys some software support (in iCOM [10]), hence might serve as a unifying paradigm – provided sufficient support of the part-whole relation will be sorted out.

In short, addressing and representing part-whole relations have several advantages, including achieving a better understanding of reality, representing it more accurately, and therefore develop better ontologies and conceptual models that in turn improve software quality (and then you have happier users & customers). If the effort that still has to be put in to meet the set of requirements is worthwhile the investment of resources, that will only be known when it is tried and tested.

6 Concluding remarks

We looked at part-whole relations from several perspectives: philosophical considerations, mathematical properties, variations on types of part-whole relations, and its (non-) treatment in several knowledge representation and conceptual data modelling languages. Each sub-topic covered only scratched the surface of the characteristics of the part-whole relations within their respective fields of research. Coherently linking up these approaches remains a distant goal, although some links are being established.

To give an indication of open problems, we outline several topics, although this list is not exhaustive.

- ⋆ Philosophy: What are the precise differences between parthood relations between classes and parthood relations between instances? How can the issues around the relata of the parthood relation be resolved? What about the inverse relation *has_part* and inter-part relations? How do mereology and granularity relate? Concerning extensions as discussed and proposed for UML, what are the consequences of so-called secondary properties for mereology?
- ⋆ Mathematical open issues: a full characterization of models of **GEM**, uncovering which properties distinguish models of **GEM** that are not models of **GEM+** from models of **GEM+**, and the model-theoretic analysis of the various topological and other extensions of GEM [30].
- ⋆ DL: Which subtheory of mereology fits best with any of the extant DLs? Or can one of the DL languages be extended, and if so, how and what about its complexity? How does it compare with SEP triples in domain modelling? What about property inheritance across the parthood relation, and about essential, mandatory, and shared parts? What are the difference between the intensional and extensional reasoning (behaviour of the parthood relation at the TBox and ABox, respectively)?

⋆ ER and ORM: what and how to add it? What about developing more expressive versions of ER and ORM that include the parthood relation, alike the different DL languages? How to make it usable for the modeler?
⋆ UML: Which extensions have been proposed, where do they differ and where does the same aspect have alternative but equivalent representations? What are the exact differences with mereology proper and the engineering requirements, and why? How can it be transferred to the application stage and to make is usable for the modeler?
⋆ RDF and XML: Little has been investigated. It is my guesstimate that proper treatment of the parthood relation in XML does not seem likely as it is not expressive regarding relations in general, let alone deal with a range of properties of a relation. I would like to see my guesstimate to be proven incorrect.
⋆ Applied parthood relations: what lessons can be learned from using part-whole relations in specific subject domains such as bio-ontologies and geographical information systems? If any, can this be fed back into mereology to extend mereological theories? Does usage of the part-whole relation across subject domains reveal domain specific intricacies that cannot be generalised to domain-independent characteristics?
⋆ How can the 'chain' from philosophy to mathematics to conceptual modelling be linked up in a consistent and coherent manner?
⋆ Can an ontologically correct taxonomy of types of part-whole relations be constructed such that it complies with at least one of the foundational ontologies?

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Note: main references (regarding relative importance, novelty, comprehensiveness and/or clarity) are indicated in bold face. The list is incomplete; check the bibliographies of the main references for many more publications.